Professor Hermann Albert Kobold, Director of the Centralstelle, Kiel, Germany,

was proposed by the Council as an Associate of the Society.

Seventy-two presents were announced as having been received since the last meeting, including, amongst others:—

T. W. Backhouse, Catalogue of 9842 Stars, or all Stars very conspicuous to the naked eye for the epoch of 1900; and E. W. Maunder, the Science of the Stars, presented by the authors.

Harvard Observatory Annals, vols. lvi., No. 4, and lxi., part 3 (Mrs. Fleming, stars having peculiar spectra, and W. H. Pickering, Statistical investigation of Cometary Orbits), presented by the Observatory.

On a Device for Facilitating Harmonic Analysis and Synthesis. (Plate 5.) By Ernest W. Brown, M.A., Sc.D., F.R.S.

1. It has been mentioned in previous papers that some mechanical device was much needed for dealing with the large number of small harmonic terms which are present in the expressions for the moon's place. Several plans for doing this work mechanically have been drawn up and rejected, chiefly because they required the attendance of either a skilled artisan or a skilled computer. The problem, briefly stated, consists of two parts. One is the arrangement of the large masses of numbers which have to be added in different ways; the other is the actual addition of these numbers.

2. The introduction of the comptometer has made the second part of the problem quite simple. No matter how long the column of figures, the addition can be done rapidly and accurately without concentrated attention. Unless it be desired to print every number as it is added to the total, those forms of adding machines in which the simple pressure on a key performs both the addition of the digit and the "carrying over" are more economical of time than those machines in which a lever must be pulled to do this work after the proper keys have been depressed. The point is not of importance when each number consists of many digits, but it makes a considerable difference when there are many numbers of from one to three digits each. The chief saving of time in the use of a machine, however, is gained by an operator who can strike the correct keys with his right hand without looking at the machine, so that his attention is not distracted from the column of figures he is following with his left hand. In order to do this easily, he is frequently recommended not to use the keys 6, 7, 8, 9, but to add those digits by

striking $3+3$, $3+4$, $4+4$, $4+5$, respectively. Further assistance is afforded by the construction of the faces of the keys; the even numbers have flat faces and the odd hollow faces, and the sense of touch is of great assistance in avoiding errors. All the skill needed is learnt in three or four weeks, so that the expense of operation is small.

3. The difficult part of the problem is the arrangement of the numbers to be added so that the operator may not need constant direction and is not required to spend a considerable part of his time over it. The device described here has been made for this purpose and is now in use. It consists of tapes on which the numbers to be added are written, a carrier which holds these tapes, and a frame on which the carrier is placed, with guides to prevent the tapes from becoming entangled or getting out of position. The dimensions of the various parts can be varied at will. Those adopted and described here were considered to be best suited to the particular work in hand. No part requires any special accuracy of construction.

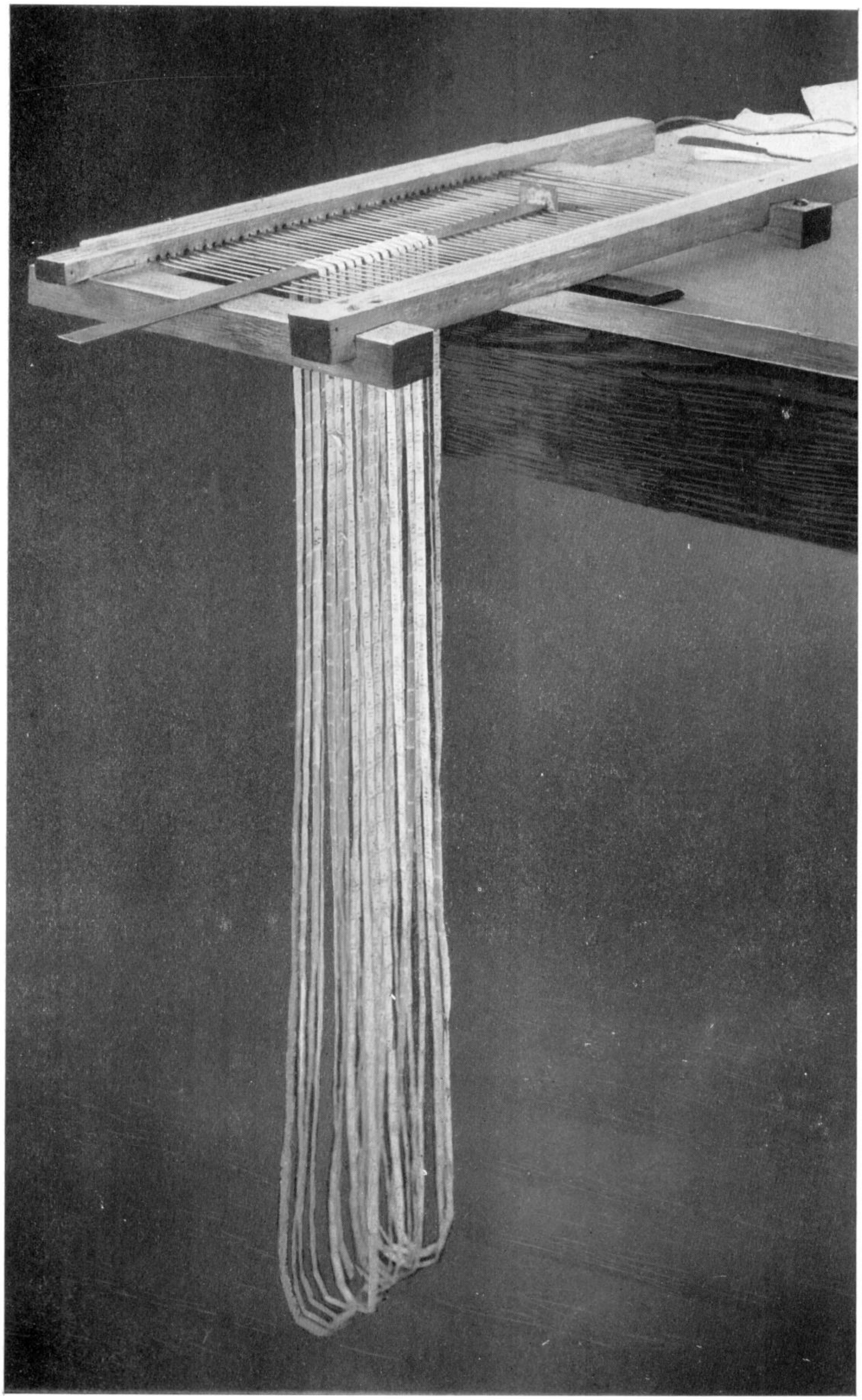
The "tapes" consist of rectangles of cardboard $\frac{3}{16}$ in. by $\frac{3}{4}$ in. pasted lengthwise on a heavy cambric tape, $\frac{1}{16}$ in. being left between the ends of adjacent cards. The width of the tapes is $\frac{3}{16}$ in., and various lengths are necessary according to the work in hand.

The frame is rectangular, and consists of four pieces of oak of cross section 1 in. by 1 in., two of them being 24 ins. long and the other two 12 ins. The illustration (Plate 5) shows the method of joining them together, the shorter pieces being let into the longer ones, which are 7·5 in. apart, to a depth of $\frac{1}{2}$ in., and fastened in position by screws.*

Before the frame was put together the middle of each side was bored with 47 (an arbitrary number) holes $\frac{5}{16}$ in. apart, the holes on one side being made to a depth of $\frac{1}{2}$ in. and on the other side right through. After the frame had been screwed together, the guides (ordinary knitting needles 8 in. long) were inserted through the open holes and pushed through into the opposite holes. They are prevented from falling out by a piece of wood $\frac{1}{2}$ in. by $\frac{1}{4}$ in. cross section, extending the whole length of the outer side of the frame so as to cover all the outer ends of the open holes, and fastened with screws. The holes were so drilled that the guides are not tightly held. The frame is screwed on to a table with one end near the edge in order that the portion carrying the guides may project over the edge.

The carrier is a flat piece of brass $\frac{1}{16}$ in. by $\frac{3}{4}$ in. cross section and about 18 in. long. On one end, perpendicular to its length, a piece of brass about 1·5 in. square and of small thickness was soldered, so that when the carrier is laid on the frame it will not slip across the guides.

* The frame at present in use was made by the writer. The tapes were made to order by the firm of Bradley & Scoville of New Haven, and cost from one to two dollars per length of 1000 cards.



A DEVICE FOR FACILITATING HARMONIC ANALYSIS AND SYNTHESIS.
ERNEST W. BROWN.

The illustration shows the device arranged for the formation of a double-entry table (see No. 13). The method of using it is shown in the various applications described below.

The Summation of Harmonic Terms at Equal Intervals of Time.

4. In the summation of harmonic terms at numerous dates we can always limit the number of values of any one term which need be calculated. This number is such that the maximum difference between two consecutive values shall be less than the pre-assigned error which is to be allowed. The intervals at which the term is to be computed are not usually in an exact ratio to the period of the term. Suppose that the interval between the successive values required is taken to be unity, and let there be a of these intervals in the period of the term. The period a is not a whole or fractional number in general; let it be expressed by a convergent p/q. If we divide the argument into p parts and let it run through q periods, every value corresponding to each part will occur once and only once. Hence the convergent must be so chosen that this number, p, of parts shall give all the values of the function to the required degree of accuracy. In the choice of the most suitable convergent, the value of q should be as small as is consistent with this condition. The unit interval consists of q of the p parts into which the argument is divided. Since p and q are prime to one another, if we start at any one of the p values and take the values successively at unit intervals, that is, at every q parts of the argument, all the p values will occur and we shall run through q periods of the term before returning to the initial value.

5. Use is made of these facts by taking a length of the tape containing p cards and joining the ends. The p values of the function are written on successive cards of the tape in the order in which they occur when unit intervals are taken. Suppose that the convergent is 47/15. After computing the 47 values, we write them on the cards of the tape in the order 0, 15, 30, 45, 13 (=60–47), 28, 43, 11 (=58–47), 26, . . . Thus, wherever we start on the tape the following card always gives the value of the term one unit interval later. Every term to be included is treated in this way. The unit interval is the same for all, but the periods and convergents will differ.

After each term has been entered on its tape the tapes are placed over the carrier, the hanging portions dropped between the guides, and the carrier is then lowered until it rests on the frame. One card of each tape then rests on the face of the carrier.

The argument of each term for the first date required is computed. As there will generally be more than one complete period on each tape, that number amongst the arguments on the tape must be chosen which gives a value of the function nearest to the value at the given date. The tape is then moved over

the carrier (which is meanwhile held fixed) until the proper argument appears on its face.

After all the tapes have been thus adjusted the numbers are ready for summation. The sum of those appearing on the face of the carrier is the sum of the harmonic terms at the first date. This is found on the comptometer and recorded. The carrier is then turned through 180°. This turn brings the next succeeding card of each tape on to the face of the carrier. According to the process of formation described above, the numbers so appearing are the values of the terms a unit interval later. They are summed and recorded. The process is continued until a certain number of sums has been obtained.

6. Since the period used is only an approximation to the actual period, there will be an error which will gradually accumulate in each term, so that after a number of turns of the carrier the cards on its face will no longer correspond to the date which has been reached. With a given number of turns the amount of the error in any term will depend on the convergent adopted, and will be less the closer the convergent is to the actual period. Hence the number q of periods on a tape should be as great as is possible without making the tapes of inconvenient length. At a date when the error becomes sensible, the tapes must be readjusted. This requires that the arguments for this date be computed with the exact period. Suppose that 50 consecutive sums can always be obtained with the required accuracy by simply turning the carrier. The operator must be given the new arguments to be used for each tape after every 50 intervals. For 5000 sums he will need 100 arguments for each term. These are all given to him at the start of the work, so that he may do the whole of it without supervision. The formation of these arguments is not difficult and can be done mechanically on an arithmometer.

The readjustment to the new arguments has been found to require but little time or trouble. The results can be tested by differences and by duplicate calculation. If the latter is done backwards with the same set of arguments, most of any accumulated error will disappear on taking the mean between the two values for the same date obtained in the two calculations.

7. Various modifications to deal with special cases may be adopted. The frame is wide enough to admit of at least four carriers side by side. When some of the periods are very long, the corresponding tapes would be of an inconvenient length. We can take an integral number l of time-intervals as our unit for these. The convergents to periods consisting of a time-intervals are then found for the number a/l instead of for a. The carrier with these long-period terms must be turned once to every l turns of the other carrier. The signal for turning may be given by an extra tape on the latter carrier.

Sometimes the convergents are inconvenient, especially when one gives too small a value of p and the next so large a value

that the tape will be inconveniently long. This difficulty has been overcome in some cases by a device suggested by Mr. Hedrick, who has also greatly assisted in working out all the details in the methods of this paper. The term is computed with the smaller convergent but with half the given amplitude, and the numbers are written on a tape in the usual manner. The same series is copied on another tape. If we add the numbers on the two tapes having the same argument we get the value of the term for one of the p parts; if we displace one of the tapes so that the p^{th} part on one is added to the $(p+1)^{\text{th}}$ part on the other we obtain a value of the term which is half way between these parts.* Of course three tapes may be used in this way if necessary, so as to secure a division of the circumference into $3p$ parts.

In one case two successive convergents 12/1, 743/62 were found. The convergent 12/1 will permit of a run for 50 successive sums, but the amplitude was such that p should be about 120. The difficulty has been solved by using two tapes with periods having convergents 119/10, 121/10, respectively, and with amplitudes equal to half the amplitude of the term. These can always be so adjusted that the sum of the two values appearing on the face of the carrier is the correct value of the term for a run of 50 sums, within the limits of error allowed.

8. If we need the sum of the sines as well as that of the cosines, two carriers may be placed side by side on the frame, each tape passing over both of them. A tape is drawn up and a loop of it dropped between the carriers. This loop is adjusted to correspond to one quarter of the period of each term. Each tape is treated in the same way. The numbers on one carrier when added give the sum of the cosines; those on the other, the sum of the sines. To get these sums for the next date each carrier is turned through 180°.

The Formation of a Double-Entry Table.

9. Suppose that it be required to form a double-entry table of the terms †

$$S = \Sigma_{i,j} A_{i,j}\{1 + \cos(i\theta + j\phi + \alpha_{i,j})\},$$

where i, j are integers, θ, ϕ variable arguments, and A, α constants, the same or different for each term. A specific case will most easily explain the method. Suppose that a division of θ into 73 parts and of ϕ into 73 parts gives a sufficient number of values of S for the purposes required. What we need is the value of S when θ, ϕ are any multiples of 360°/73. Put d for 360°/73. The

* Provided second differences be negligible.

† Unity is added to every sine or cosine so that the function to be treated shall be always positive. This plan is always adopted. In the analysis of observations, considered below, the same number is added to every observation for the same purpose.

method consists in finding S for $\phi = 0$ when $\theta = 0, d, 2d, \ldots 72d$; then for $\phi = d$, $\theta =$ same series; and so on up to $\phi = 72d$.

Since i, j are integers, and since $id + jd =$ multiple of d when multiples of $360°$ are rejected, it is evident that we only require the values of each term for the angles $i\theta + j\phi = 0, d, 2d, \ldots 72d$. Hence

$$A_{i,j}\{1 + \cos(p.d + a_{i,j})\}$$

is first computed for each term with $p = 0, 1, 2, \ldots, 72$.

10. Suppose $i = 2$, $j = 3$. Let the 73 values of the term be entered on a tape containing 73 cards with its ends joined together, in the order $p = 0, 2, 4, \ldots 72, 74 (= 1), 3, 5, \ldots 71$. Number them in red ink 0 to 72 in the order in which they are written. Then one turn of the carrier is equivalent to increasing θ by d. If we start with the red number 0, we shall have the value corresponding to $\phi = 0$, and by turning the carrier we shall see the successive values for $\phi = d, 2d, \ldots 72d$.

When $\phi = d$, we have $j\phi = 3d$. We must therefore start with the red number corresponding to $3d$, that is, the number 38, and then successive turns give the values for $\theta = d, 2d, \ldots 72d$ with $\phi = d$. Similarly for all the values of ϕ.

The formation of these starting arguments is simple. For

$$\phi = 0, d, 2d, \ldots$$

we have

$$3\phi = 0, 3d, 6d, \ldots 72d, 2d, \ldots 71d, d, \ldots 70d,$$

giving the card numbers

$$0, 38, 3, 41, 6, 44, \ldots$$

that is, the remainders after the successive multiples of 38 have been divided by 73.

11. Tapes are made out for all the terms in a similar manner, the number 73 being the same for all. They are placed over the carrier, which is lowered on to the frame so that the hanging portions of the tapes fall between the guides as before. For $\phi = 0$, $\theta = 0$, all the card numbers are 0 and these are brought on to the face of the carrier. The 73 sums obtained after each turn of the carrier are the values of S for $\phi = 0$, $\theta = 0, d, \ldots 72d$. The tapes are then readjusted for the value $\phi = d$, $\theta = 0$, and the same process followed. The 73 card numbers required for each term at the beginning of each set of 73 sums are given to the operator at the start, so that he can form all the 73^2 sums without further supervision.

In practice some modifications have been made in the 21 tables computed in this way. It was not necessary in some of the tables to obtain S for all the values of ϕ; alternate values were then computed. It is evident also that all the terms for which i/j has the same value may be put together on one tape.

If there is a term for which $i=0$, the tape for it is held fixed during each series of 73 sums. After the table of 73^2 sums has been completed, some interpolation and rearrangement are necessary to put it into the usual form required in lunar tables; this, however, is another question which need not be insisted on here.

12. The need for the number of parts into which θ, ϕ are divided to be prime to i and j is easily seen. For example, if we had used 68 instead of 73, the first 34 turns of the carrier in the term for $i=2, j=3$, would bring us back to zero. Thus only 34 values would be required for $\phi=0$. When, however, $\phi=360°/68$ the proper values would not appear on the tape at all. Similarly if the number is not prime to j.

13. The illustration exhibits the device arranged for this purpose. There are 13 tapes corresponding to the 13 terms which the double-entry table under treatment contains. The operator sits in front with the frame on his left so that he may follow the numbers on the carrier with his left hand while he operates the comptometer with his right hand. Also on the left is a Burroughs adding and printing machine on which each sum as it is obtained is recorded. This machine also records the sum of each series of p ($=73$ in this case) sums. Since every value on each tape is used once and only once in each series, the sum of each series of sums should be always the same (unless there is a term for which $i=0$). This furnishes a useful test. Errors are detected also by examining the differences. Duplicate calculation has not been necessary.

Analysis of Observations.

14. When the analysis of numerous observations at equal time-intervals into many harmonic constituents is required, as in tidal work, the labour of direction and computation can be reduced to equally simple rules. Suppose, for example, a year's observations of tide heights at hourly intervals is given * and the usual process for the analysis of observations be adopted. There will be nearly 8800 observations, and the work consists in forming sums in groups of these heights in as many different ways as there are harmonic terms to be found. The number of the sums to be obtained varies with the period of the term, but practically all the observations are used in every term. With the comptometer the actual work in the case of each term is the same, for there is no difference between the labour of finding 24 sums each of 350 numbers and that of finding 12 sums each of 700 numbers.

* This example is taken only for the purpose of illustrating the use of the device for the analysis of large numbers of observations. An attempt to discover whether there is any saving of expense in this special problem over that necessary with the use of the apparatus described by Sir George Darwin in *Proc. Roy. Soc.*, vol. 52 (1892), p. 345, would require a careful analysis and probably also a reconstruction of the general plan of reduction. I am much indebted to Sir George Darwin for criticisms on the method of presentation of the device and its applications, which led to this paper being almost entirely rewritten.

Let the observations be written on a tape long enough to contain them all, the successive cards being numbered 0, 1, 2, . . . preferably in red ink. The ends of the tape are not joined together. Hence, in order to prevent the tape from slipping off the carrier when any one of the first few or of the last few numbers appear on its face, about 20 cards at the beginning and at the end of the tape should be left blank. About 600 feet of it will be required.

Suppose that it is desired to analyse for a period of $25^{h}\ 49^{m}$. For this we obtain the average of the heights at the exact hours which are nearest to multiples of $25^{h}\ 49^{m}$, that is, at 0^{h}, 26^{h}, 51^{h}, 77^{h}, . . .,* then those an hour later in each case, and so on up to those which are 25 hours later. For this arrangement, the tape is wrapped over and over the carrier, not tightly but with long loops hanging down, there being 24 or 25 cards on each loop. There will then be some 350 cards on the face of the carrier and 350 loops. The operator is instructed to adjust the tape so that the red numbers 0, 26, 51, 77, . . . appear on the face of the carrier. No further instructions, except that of adding the observations thus presented, are necessary. After this is done and the sum recorded, he turns the carrier through 180°. This brings on to its face the observations at 1^{h}, 27^{h}, 52^{h}, 78^{h}, . . . which are similarly added and recorded. After 26 sums have been thus obtained, each sum is divided by 350 and we have the average values at 0^{h}, 1^{h}, . . . 25^{h}, of the harmonic term of period $25^{h}\ 49'$. These are analysed in the usual way in order to find the amplitude and epoch.

15. As one carrier bearing 350 loops is not a convenient arrangement the following modifications may be made in order to bring the work within the scope of a frame but little larger than that described above. If we analyse for a period of $3 \times 25^{h}\ 49^{m} = 77^{h}\ 27^{m}$, we really obtain the ordinates of the Fourier curve for

$$\Sigma A_i \cos(i\theta + \alpha_i) \qquad i = 0, 1, 2, \ldots,$$

where the period of θ is $77^{h}\ 27^{m}$. If there are no terms present except that of $i=3$, the curve of averages will actually consist of ordinates for three successive periods of the term $A_3 \cos(3\theta + \alpha_3)$ which has a period of $25^{h}\ 49^{m}$. The first period starts at 0^{h}, the second at $25^{h}\ 49^{m}$, and the third at $51^{h}\ 38^{m}$. The ordinates between 25^{h} and 52^{h} must then be interpolated so as to get the values at hourly intervals from $25^{h}\ 49^{m}$ to $51^{h}\ 38^{m}$. The ordinates between 51^{h}, 78^{h}, are treated in a similar manner. We thus have three sets of values each of which starts at the same interval from the maximum; the average gives the 26 numbers from which the amplitude and epoch are deduced as before. The tides with periods

* If the exact period be expressed in decimals of an hour these numbers can be formed rapidly on an arithmometer. They are, however, the same for all ports.

approximating to 12 hours are similarly dealt with, but here we have six periods in the range.

With this method, the tape is wrapped round the carrier as before but so as to give rather less than 120 loops, there being 76 or 77 cards in each loop. The loops will hang below the carrier to a depth of 30 ins.—sufficient to keep their lower portions off the floor. The instructions to the operator then consist of the numbers 0, 77, 155, 232, . . . of the cards which are to appear on the face of the carrier at the start. He operates as before to obtain the 78 sums.

With this modification goes another, which simply consists in arranging the tape so as to show three columns of 40 numbers each instead of one of 120 numbers. Room on the frame is provided for several carriers side by side. If we take three, the tape can go 40 times over the first carrier, then 40 times over the second, then 40 times over the third, care being taken that the directions of wrapping are clockwise (or counter-clockwise) on all three. After the sum of the three columns has been obtained all three carriers are turned through 180°.

16. The labour of making 120 turns of the tape round the three carriers can be reduced to 40 turns, as follows:—Wrap the tape round one carrier so as to make 40 loops each of 232 or 233 cards, drop the loops between the guides, and lower on to the frame. Insert two other carriers underneath that already in position. Separate the three carriers so that they lie side by side on the frame. Then draw up each loop with the hand and drop a third of it between each pair of adjacent carriers, so that it will resemble the letter W. After adjustment of the tape according to the given table of numbers, the first carrier will have on its face the observations numbered 0, 232, 465, . . .; the second, those numbered 77, 310, . . .; and the third, those numbered 155, 387, . . . The only difference is that the same 120 numbers will be summed in three columns in a different order to the previous arrangement of three columns.

17. Various modifications of the mechanical arrangement of the tape will suggest themselves. With a slightly larger frame, six carriers, each bearing about 60 loops, could be used; the average ordinate at each hour of the period approximating to 25 or 26 hours could then be obtained directly. Or the tape might be divided into a few sections and the results in each section obtained separately.

18. After the summation for one period has been performed the operator has only to readjust the tape so that the cards appearing on the faces of the carriers bear the numbers proper to the new period and proceed as before. Once the tape has been placed over the carriers the readjustment is rapidly and easily done. If, in doing so, a loop becomes so short that much moving of the tape is necessary, it may always be dropped. A spare carrier is inserted underneath that bearing the short loop and pushed up to the loop. The original carrier is then withdrawn in the opposite direction

until the ends of the two carriers are separated, when the loop can be dropped without disturbing the other portions of the tape. The original carrier is then pushed back into place and the spare one withdrawn.

19. As a loop is not easily added, the terms should be taken in the order of the length of their periods, the shortest coming first. For very long periods, the lower parts of the loops may be resting on the floor; there is no real objection to this, if care be taken that they do not become entangled. Even this can be avoided by using other carriers to hold up the loops, the cards on the face of one carrier only being used in forming the sums.

20. The actual operation of the device has so far been limited to forming double-entry tables. The arrangements and preliminary calculations for summing numerous small harmonic terms are under way, and the work of summation will be started as soon as the twenty-one double-entry tables have been finished.

It may be stated that very little trouble has been experienced by cards becoming detached from the tape. The tapes so far used have each made between one and two hundred complete revolutions over the carrier, and would probably stand a thousand before becoming too much worn for further use.

21. If dimensions different from those described above be adopted, attention should be paid to the following details. In order to avoid a tendency to force the cards off the tape, the breadth of the carrier should be slightly less than the length of a card, and the distance between consecutive cards a little greater than the thickness of the carrier, the latter difference not to be less than the former. The distance between the guides should be a little greater than the width of the tape to prevent friction. A certain amount of "play" should be allowed for the guides in their sockets so that a slight variation in either the width of the tape or the distance between the guides may not cause trouble. For offices in which work of this nature is continuous, it is possible that more permanent forms of tape might be made, say of xylonite cards joined by small rings, from which the observations might be washed after the analysis has been completed.

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